CUT-OFF FUNCTION LEMMA IN $\mathbb{P}^k$

TAEYONG AHN

Abstract. In this note, we compute a cut-off function over $\mathbb{P}^k$. Let sufficiently small $\delta > 0$ be given. When we are given a compact set K in $\mathbb{P}^k$ and a prescribed open neighborhood K_δ of K, we find a smooth cut-off function χ_δ such that $\chi_\delta \equiv 1$ over K and $supp(\chi_\delta) \subseteq K_\delta$, where K_δ denotes the set of points whose distance to K is less than δ with respect to the Fubini-Study metric of $\mathbb{P}^k$. Moreover, we estimate the bound of the derivatives of χ_δ in terms of δ. It seems to be well-known, but we want to provide detailed computations. They are very elementary.

1. Introduction

In this note, our space is $\mathbb{P}^k$ and we assume that the distance is measured with respect to the Fubini-Study metric if we do not specify.

Let $\delta_0 > 0$ be given. We consider $0 < \delta < \delta_0$. Let $K \subseteq \mathbb{P}^k$ be compact and K_δ a δ-neighborhood of K, that is, the set of points whose distance to K is less than δ with respect to the Fubini-Study metric. We want to prove the following lemma:

Lemma 1.1. *There exists a smooth cut-off function $\chi_\delta : \mathbb{P}^k \to [0,1]$ such that $\chi_\delta \equiv 1$ over K and $supp(\chi_\delta) \subseteq K_\delta$. Moreover, $\|\chi_\delta\|_{C^\alpha} \lesssim |\delta|^{-\alpha}$ as δ varies.*

Here, $\|\cdot\|_{C^\alpha}$ denotes the C^α-norm of the function. The idea is simply to smooth out a characteristic function by convolution (of the Lie group of automorphisms over $\mathbb{P}^k$).

2. Family of Local Coordinate Charts of $\mathbb{P}^k$

It suffices to prove the lemma for a fixed family of local coordinate charts. Thus, we will fix one as follows.

For $\mathbb{P}^k$, we can find k natural affine coordinate charts covering $\mathbb{P}^k$ of the form $\{[z_0 : ... : z_{i-1} : 1 : z_{i+1} : ... : z_k] | z_j \in \mathbb{C} \text{ for } j \neq i\}$ for $i = 0, ..., k$, which we will call the Z_i-coordinate chart. For this chart, there is a natural coordinate map $\zeta_i : Z_i \to \mathbb{C}^i \times \{1\} \times \mathbb{C}^{k-i}$ defined by $\zeta_i([z_0 : ... : z_{i-1} : 1 : z_{i+1} : ... : z_k]) = (z_0, ..., z_{i-1}, 1, z_{i+1}, ..., z_k)$.

We defined a norm $\|\cdot\|_i$ defined by

$$\|(z_0, ..., z_{i-1}, 1, z_{i+1}, ..., z_k)\|_i = (|z_0|^2 + ... + |z_{i-1}|^2 + |z_{i+1}|^2 + ... + |z_k|^2)^{\frac{1}{2}}$$

for each $\mathbb{C}^i \times \{1\} \times \mathbb{C}^{k-i}$.

1991 *Mathematics Subject Classification.* TBA.
Key words and phrases. TBA.
TBA.

3. Automorphism group of $\mathbb{P}^k$

The group $\mathrm{Aut}(\mathbb{P}^{\mathrm{k}}) = \mathrm{PGL}(\mathrm{k}+1,\mathbb{C})$ of automorphisms of $\mathbb{P}^k$ is a complex Lie group of complex dimension k^2+2k. An element of $\mathrm{Aut}(\mathbb{P}^{\mathrm{k}})$ can be understood as an equivalence class of the complex $(k+1)\times(k+1)$ matrix group under the equivalence relation given by scaling.

Without loss of generality, we may consider a point $z \in Z_0$ and its coordinates $\zeta \in \{1\}\times\mathbb{C}^k$. Let $h = (0, h_1, h_2, ..., h_k)$ with $|h_i| < \epsilon$ for sufficiently small $\epsilon > 0$. Then $\zeta + h \in \{1\}\times\mathbb{C}^k$ is a very close point near $\zeta \in \{1\}\times\mathbb{C}^k$, where the addition is coordinatewise and we can find a unique linear map $G_h : \{1\}\times\mathbb{C}^k \to \{1\}\times\mathbb{C}^k$ defined by

$$G_h = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ h_1 & 1 & 0 & \cdots & 0 \\ h_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_n & 0 & 0 & \cdots & 1 \end{pmatrix}$$

such that $G_h(\zeta) = \zeta + h$. Note that $G_h \circ G_{-h} = G_{-h}\circ G_h = Id$.

Using the exponential map of Lie algebra to Lie group, we can find holomorphic coordinates $\psi : sl(k+1,\mathbb{C}) \to PGL(k+1,\mathbb{C})$ near $Id \in PGL(k+1,\mathbb{C})$ where $sl(k+1,\mathbb{C})$ is the special linear Lie algebra, which is the set of $(k+1)\times(k+1)$ matrices with zero trace. Near the $Id \in PGL(k+1,\mathbb{C})$, we can also find a representation $PGL(k+1,\mathbb{C}) \to GL(k+1,\mathbb{C})$ by picking a $(k+1)\times(k+1)$ matrix with the $(1,1)$-component being 1. Let ϕ denote this representation. We consider the following diagram

$$\begin{array}{ccc} sl(k+1,\mathbb{C}) & \longrightarrow^{H_h} & sl(k+1,\mathbb{C}) \\ \psi\downarrow & & \psi\downarrow \\ PGL(k+1,\mathbb{C}) & \longrightarrow^{[G_h]} & PGL(k+1,\mathbb{C}) \\ \phi\downarrow & & \phi\downarrow \\ GL(k+1,\mathbb{C}) & \longrightarrow^{\overline{G_h}} & GL(k+1,\mathbb{C}), \end{array}$$

where in the second line, $[\cdot]$ means the equivalence class that contains the inside element, $[G_h][A] = [A\cdot G_h]$ for $[A] \in PGL(k+1,\mathbb{C})$, and $\overline{G_h}$ is defined as follows:

$$\begin{pmatrix} 1 & a_{1,2} & \cdots & a_{1,k+1} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,k+1} \\ a_{3,1} & a_{3,2} & \cdots & a_{3,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+1,1} & a_{k+1,2} & \cdots & a_{k+1,k+1} \end{pmatrix}$$

$$\downarrow \overline{G_h}$$

$$\frac{1}{1+\sum_{i=2}^{k+1} a_{2,i}\cdot h_{i-1}} \begin{pmatrix} 1 & a_{1,2} & \cdots & a_{1,k+1} \\ a_{2,1} + \sum_{i=2}^{k+1} a_{2,i}\cdot h_{i-1} & a_{2,2} & \cdots & a_{2,k+1} \\ a_{3,1} + \sum_{i=2}^{k+1} a_{3,i}\cdot h_{i-1} & a_{3,2} & \cdots & a_{3,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+1,1} + \sum_{i=2}^{k+1} a_{k+1,i}\cdot h_{i-1} & a_{k+1,2} & \cdots & a_{k+1,k+1} \end{pmatrix}.$$

Note that $H_h, [G_h]$ and $\overline{G_h}$ in the diagram are not defined over the entire space. However, there exists a sufficiently small $\epsilon > 0$ such that for all $\{h_i\}_{i=1}^n$ with $|h_i| < \epsilon$ for $i = 1, ..., n$, $\overline{G_h}$ is well-defined over all $A \in GL(n+1, \mathbb{C})$ with $\|A - Id\| < \epsilon$ and with the $(1,1)$-component of A being 1, where $\|\cdot\|$ is the standard matrix norm. Since ϕ and ψ are local biholomorphisms, we can also find correspoinding subsets in $PGL(k+1, \mathbb{C})$ and $sl(k+1, \mathbb{C})$.

We identify $sl(k+1, \mathbb{C})$ with $\mathbb{C}^{k^2+2k}$ and the set of representations of $PGL(k+1, \mathbb{C})$ with $\mathbb{C}^{k^2+2k}$. For convenience, we use $x = (x_1, ..., x_{k^2+2k})$ for $sl(k+1, \mathbb{C})$ and $\xi = (\xi_1, ..., \xi_{k^2+2k})$ for the other. Then

$$\begin{array}{ccc} \mathbb{C}^{k^2+2k} & \longrightarrow^{H_h} & \mathbb{C}^{k^2+2k} \\ \psi \downarrow & & \psi \downarrow \\ PGL(k+1, \mathbb{C}) & \longrightarrow^{[G_h]} & PGL(k+1, \mathbb{C}) \\ \phi \downarrow & & \phi \downarrow \\ \mathbb{C}^{k^2+2k} & \longrightarrow^{\overline{G_h}} & \mathbb{C}^{k^2+2k}. \end{array}$$

We denote $\phi \circ \psi$ by Φ. Then, $\xi_i = \Phi_i(x_1, ..., x_{k^2+2k})$ for $i = 1, ..., k^2+2k$ and the map $H_h = \Phi^{-1} \circ \overline{G_h} \circ \Phi$ is a map from $\mathbb{C}^{k^2+2k}$ to $\mathbb{C}^{k^2+2k}$. Note that in our case, ψ, ϕ are smooth and $\overline{G_h}$ is smooth with respect to h.

4. Measures on $sl(k+1, \mathbb{C})$

Recall that x is used for $sl(k+1, \mathbb{C})$. Let λ denote the standard Euclidean measure on $sl(k+1, \mathbb{C})$. We assign the standard matrix norm $\|x\|_s$ to each $x \in sl(k+1, \mathbb{C})$. We consider a smooth radial probability measure μ over the coordinate $sl(k+1, \mathbb{C})$ centered at $O \in sl(k+1, \mathbb{C})$ with its support $\|x\|_s < \sigma$ for sufficiently small $\sigma > 0$, which makes $\Phi(\{\|x\|_s < \sigma\}) \subseteq \{\|A - Id\| < \epsilon\}$. Then, $d\mu = M(x)d\lambda$ where M is a smooth function defined on $sl(k+1, \mathbb{C})$ and has support in $\|x\|_s < \sigma$.

Let $h_\theta : sl(k+1, \mathbb{C}) \longrightarrow sl(k+1, \mathbb{C})$ be a scaling map by θ for $|\theta| \leq 1$. We define $\mu_\theta := (h_\theta)_*(\mu)$. Then, μ_θ is a smooth measure for $\theta \neq 0$ and a Dirac measure at $O \in sl(k+1, \mathbb{C})$ for $\theta = 0$. Note that the support of μ is in $\{\|x\|_s \leq \theta\sigma\} \subseteq \{\|x\|_s \leq \sigma\}$.

For the better terminology, by the derivatives of μ_θ, we mean the derivatives of the Radon-Nikodym derivative of μ_θ with respect to the standard Euclidean measure λ.

5. Regularization

In this section, we define a regularization of a bounded function and provide the estimate of the regularity.

Let f be a bounded complex-valued function over $\mathbb{P}^k$ with compact support. Without loss of generality, we may assume that $0 \leq |f| \leq 1$. Then, we define the θ-regularization f_θ of f as being

$$f_\theta(z) = \int_{\mathrm{Aut}(\mathbb{P}^k)} ((\tau_x)_* f)(z) d\mu_\theta(x).$$

Without loss of generality, we may assume that $z \in Z_0$. Let $\zeta \in \{1\} \times \mathbb{C}^k$ be the coordinates of z and F the representation of f with respect to $\{1\} \times \mathbb{C}^k$. With respect to the coordinate $\{1\} \times \mathbb{C}^k$, we have the following representation:

$$\begin{aligned} F_\theta(\zeta + h) &= \int_{sl(k+1,\mathbb{C})} ((\Phi(x))_* F)(G_h(\zeta)) d\mu_\theta(x) \\ &= \int_{sl(k+1,\mathbb{C})} ((\Phi(H_h(x)))_* F)(\zeta) d\mu_\theta(x) \end{aligned}$$

Note that H_h is holomorphic and injective over the support of the measure μ_θ. By change of coordinates, we have

$$\begin{aligned} F_\theta(\zeta + h) &= \int_{sl(k+1,\mathbb{C})} ((\Phi(H_h(x)))_* F)(\zeta) d\mu_\theta(x) \\ &= \int_{sl(k+1,\mathbb{C})} ((\Phi(x))_* F)(\zeta)((H_h)_* d\mu_\theta)(x). \end{aligned}$$

With ζ fixed, the differentiation of the right hand side with respect to h_i's makes sense since the measure is smooth. By the direct application of the definition of the derivative, the partial derivative of $F_\theta(\zeta)$ with respect to ζ_i at ζ is the same as the partial derivative of $F_\theta(\zeta + h)$ with respect to h_i at 0. Thus, we can see that F_θ is smooth. Moreover, we can estimate its regularity.

The C^α-norm of F_θ completely depends on the value of F near ζ and the derivatives of the measure with respect to h. It is not hard to see that $(H_h)_*[(h_\theta)_* d\lambda] = |\theta|^{-2k^2-4k} d\lambda$. Indeed, Φ is a coordinate change map and G_h is a linear shear map. Thus, it remains to estimate the C^α-norm of M. So, since $(H_h)_*[(h_\theta)_* M] = M(\frac{1}{\theta}[\Phi^{-1} \circ \overline{G_h} \circ \Phi])$, the C^α-norm of $(H_h)_*[(h_\theta)_* M]$ is bounded by the product of $|\theta|^{-\alpha}$ and a constant multiple of C^α-norms of M, Φ and Φ^{-1}. Note that the latter is independent of θ.

Putting all together, since F is bounded, the support of the measure is $\|x\| \le \theta\sigma$ and $dim_{\mathbb{C}} sl(k+1, \mathbb{C})$ is $k^2 + 2k$,

$$f_{\theta C^\alpha} \lesssim |\theta|^{-2k^2-4k-\alpha} |\theta|^{2k^2+4k} \|f\|_{C^\alpha} = |\theta|^{-\alpha} \|f\|_{C^\alpha}. \tag{5.1}$$

Note that it can be more precise when we estimate the absolute value at a point in terms of its neighborhood with compact closure.

6. Main Cut-off Function Lemma

We consider two kinds of open balls in $\{1\} \times \mathbb{C}^k$. One is induced from the Fubini-Study metric of $\mathbb{P}^k$ and the other is from the standard Euclidean metric $\|\cdot\|_0$. The open ball centered at $\zeta \in \{1\} \times \mathbb{C}^k$ and of radius $r > 0$ of first kind is denoted by $B_F(\zeta, r)$ and that of second kind is denoted by $B_E(\zeta, r)$. Then, by comparison of the infinitesimal versions of the two metrics, we know that $B_E(\zeta, \frac{r}{2}\|\zeta\|_0) \subseteq B_F(\zeta, r)$.

The proof of Lemma 1.1. Note that Φ is holomorphic near the closure of the neighborhood of $\{\|x\|_s < \sigma\}$, we can find a constant $C > 0$ such that $\frac{1}{C}\|\Phi(x) - Id\| < \|x\|_s < C\|\Phi(x) - Id\|$ for $\{\|x\|_s < \sigma\}$. Here, C is independent of δ and θ. Recall that $\|\Phi(x)(\zeta) - \zeta\|_0 \le \|\Phi(x) - Id\| \|\zeta\|_0$. We take a θ such that $|\theta| \le 1$ and such

that $C\theta\sigma \leq \frac{\delta_0}{4}$. Let $C' := \frac{C\theta\sigma}{\delta_0/4} \leq 1$. Then, for all $0 < \delta < \delta_0$, we take its corresponding θ to satisfy $C\theta\sigma = C'\frac{\delta}{4}$. Note that This C' is fixed with respect to θ and δ. Then, for each $0 < \delta < \delta_0$ and for its θ, we have that for $\{\|x\|_s < \sigma\}$,

$$\begin{aligned} \|\Phi(x)(\zeta) - \zeta\|_0 &\leq \|\Phi(x) - Id\| \, \|\zeta\|_0 \leq C\,\|x\|_s\,\|\zeta\|_0 \\ &\leq C\theta\sigma\,\|\zeta\|_0 = \frac{C'\delta}{2}\frac{\|\zeta\|_0}{2} \leq \frac{\delta}{2}\frac{\|\zeta\|_0}{2}. \end{aligned} \tag{6.1}$$

Consider $K \subseteq K_{\frac{\delta}{2}} \subseteq K_\delta$. Let χ_K be the characteristic function whose support is exactly $K_{\frac{\delta}{2}}$. Then $(\chi_K)_\theta$ is the desired function with the desired estimate. Indeed, the estimate is straight forward by plugging-in $C\theta\sigma = C'\frac{\delta}{4}$ into Estimate 5.1. Equation 6.1 proves the support of the function and its region over which the function is identically 1.

So far, we have considered over Z_0 only. The above argument can be directly applied to each Z_i for $i = 0, ..., k$ in the exactly same way. Indeed, we use the same measure on $\mathrm{Aut}(\mathbb{P}^{\mathrm{k}})$ and the same constants C, C' and θ to Z_i for $i = 1, ..., k$ as in the case of Z_0. Thus, we have just proved the lemma.

□

5840 East Hall, 530 church street, Ann Arbor, MI 48109-1043
E-mail address: taeyonga@umich.edu